Contextuality and quantum weirdness

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October 22, 2021

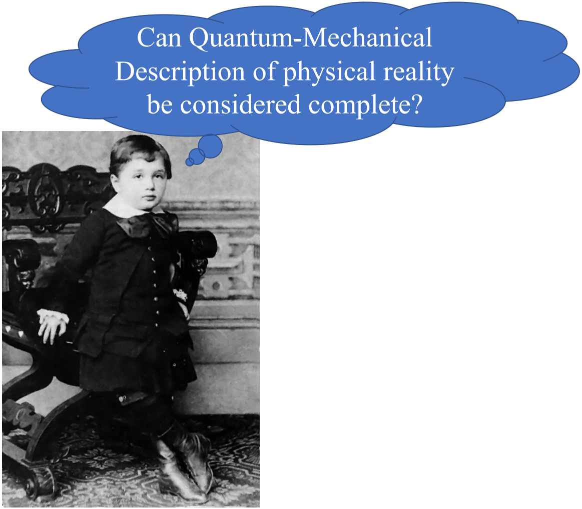
OUTLINE

INTRODUCTION

SOME HISTORY
- Hidden variable models
- Contextuality: obstacle to classical description of nature

MEASUREMENT CONTEXTUALITY
- The magic square
- Nonlocality and contextuality

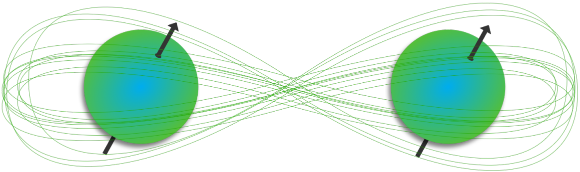
CAN WE EXPLOIT THIS?
- An application of contextuality
- A limitation of contextuality



https://en.wikipedia.org/wiki/File:Albert_Einstein_at_the_age_of_three_(1882).jpg

EPR criterion of reality:

> *"If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an **element of reality** corresponding to that quantity." (boldface added)*



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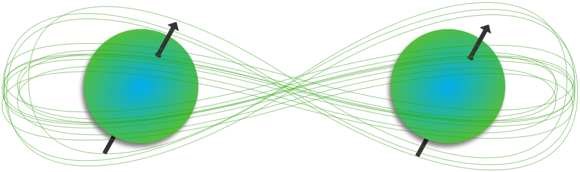
Classical attempt to describe entanglement



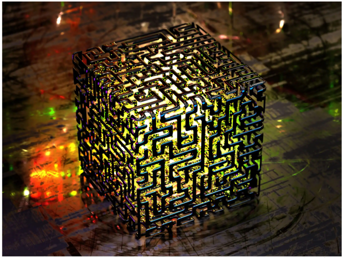
- Assign a *hidden variable* to each particle, whose values fix all measurement results for the particle.
- Then all measurements are interpreted as *revealing preexisting values* of the measured quantities.

The rise and fall of hidden variable models...

- 1935: EPR — hidden variable models (HVMs).
- 1964: Bell's theorem — $\exists$ test to distinguish quantum mechanics from local HVMs.
- 1966-67: Bell-Kochen-Specker theorem — $\exists$ test to distinguish quantum mechanics from *noncontextual* HVMs.
- 1972 onward: better and better experimental tests of Bell's theorem rule out local HVMs.
- 2000 onward: better and better experimental tests of contextuality rule out noncontextual HVMs.[1]

[1] Nice new reference: Budroni *et al.*, *Quantum Contextuality*, arXiv:2102.13036 (2021).

More about hidden variable models



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measurement #1 → $\{+1, -1\}$

measurement #2 → $\{+1, -1\}$

measurement #N → $\{+1, -1\}$

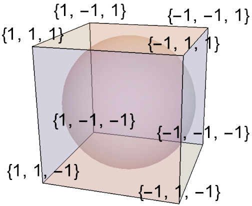
- Measurements can be compatible or incompatible.
- Measurements can have fixed relations, e.g., the product of outcomes of some three measurements might be fixed.

https://www.mouser.com/blog/Portals/11/quantumcube.JPG

More about hidden variable models

- N measurements $\Rightarrow 2^N$ joint value assignments.
- Fixed relations among measurements rule out some joint value assignments:
  - E.g., if C is the product of A and B, only possibilities are $\{+1, +1, +1\}, \{+1, -1, -1\}, \{-1, +1, -1\}, \{-1, -1, +1\}$.
- Observer's knowledge = probability distribution over joint value assignments (can reflect incompatibility).

Example:



value assignments = vertices,
prob. distributions = points in sphere.

Definition of contextuality

Contextuality: a system is *contextual* when it is impossible to construct a hidden variable model of this kind to describe it.

(Why is this called contextuality? We'll come back to that.)

But the model seemed so plausible... what could possibly go wrong?

The magic square[2]

System:

- Nine possible measurements ($A, B, C, D, E, F, G, H, I$):

A	B	C	+1
D	E	F	+1
G	H	I	+1
-1	-1	-1	

- Outcomes $= \pm 1$.
- Compatible if and only if in same row OR column.
- Product of row will always come out to $+1$, product of column to -1.

[2] N. D. Mermin, Phys. Rev. Lett. 65, 3373 (1990); A. Peres, J. Phys. A: Math. Gen. 24 L175 (1991); N. D. Mermin, Rev. Mod. Phys. 65, 803 (1993).

THE MAGIC SQUARE

A	B	C	+1
D	E	F	+1
G	H	I	+1
-1	-1	-1	

Model:

- Set of joint outcomes (the "real" states).
- Set of probability distributions (our "knowledge").

Joint outcomes: there are 2^9 if we ignore the product relations.

- How many satisfy the product relations?

THE MAGIC SQUARE

A	B	C	+1
D	E	F	+1
G	H	I	+1
-1	-1	-1	

Model:

- Set of joint outcomes (the "real" states).
- Set of probability distributions (our "knowledge").

Joint outcomes: there are 2^9 if we ignore the product relations.

- How many satisfy the product relations?

Zero.

The magic square

A	B	C	+1
D	E	F	+1
G	H	I	+1
-1	-1	-1	

No assignment of ± 1s to the magic square satisfies the product relations.

Row products $\Rightarrow$ total number of assigned -1s is even.
Column products $\Rightarrow$ total number of assigned -1s is odd.

THE MAGIC SQUARE

A	B	C	+1
D	E	F	+1
G	H	I	+1
-1	-1	-1	

No assignment of ± 1s to the magic square satisfies the product relations.

BUT...
$\exists$ physical (quantum mechanical) system whose measurements have these compatibility and product relations!

- No hidden variable model for this system is possible, and we say that it is *contextual*.

Aside: observables and Pauli operators

Quantum mechanical measurements $\sim$ operators called *observables*.

Example: Pauli matrices

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- $2 \times 2 \quad \Rightarrow \quad$ observables on two-level system = *qubit*.
- Eigenvalues $= \pm 1 \quad \Leftrightarrow \quad$ outcomes $= \pm 1$.
- Pairs anticommute, e.g., $XY = -YX$.
- $XY = iZ, \quad YZ = iX, \quad ZX = iY.$

Aside: observables and Pauli operators

Two qubits $\Rightarrow$ tensor products of Pauli matrices and 2×2 identity I form 16 *Pauli operators*

$$I \otimes I, \quad I \otimes X, \quad X \otimes X, \quad X \otimes Y, \ldots$$

Pauli matrices anticommute $\Rightarrow$ Pauli operators either commute or anticommute, e.g.,

$$(I \otimes X)(X \otimes X) = X \otimes (X^2) = X \otimes I = (X \otimes X)(I \otimes X),$$

$$(I \otimes Z)(X \otimes X) = X \otimes (iY) = i(X \otimes Y) = -(X \otimes X)(I \otimes Z),$$

$$(Z \otimes Z)(X \otimes X) = (iY) \otimes (iY) = -(Y \otimes Y) = (X \otimes X)(Z \otimes Z).$$

Realizing the magic square

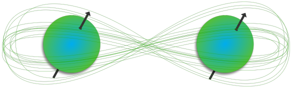
$X \otimes I$	$I \otimes X$	$X \otimes X$	+1
$I \otimes Z$	$Z \otimes I$	$Z \otimes Z$	+1
$-X \otimes Z$	$-Z \otimes X$	$Y \otimes Y$	+1
-1	-1	-1	

For example, product of bottom row is $+1 = +I \otimes I$:

$$(-X \otimes Z)(-Z \otimes X) = (XZ) \otimes (ZX) = (-iY) \otimes (iY) = Y \otimes Y$$

$$\Rightarrow \quad \underbrace{(-X \otimes Z)(-Z \otimes X)}_{Y \otimes}(Y \otimes Y) = (Y \otimes Y)(Y \otimes Y) = I \otimes I.$$

Nonlocality and contextuality



versus

?	?	?	+1
?	?	?	+1
?	?	?	+1
-1	-1	-1	

- Bell experiments rule out nonlocal HVMs; magic square (and similar) experiments rule out noncontextual HVMs.
- Magic square doesn't rely on no communication between qubits.
- Magic square is independent of quantum state.

Why "contextuality"?

?	?	?	+1
?	?	?	+1
?	?	?	+1
-1	-1	-1	

- *Noncontextual* HVM: assigned outcomes don't depend on *context*, where....
- the *context* for the measurement refers to which other compatible measurements are performed.

AN APPLICATION OF CONTEXTUALITY

Variational quantum eigensolver: given Hamiltonian

$$H = \sum_P h_P P,$$

where P are Pauli operators on n qubits and h_P are real coefficients, **estimate ground state energy**.

Method:

1. Prepare "guess" state ("*ansatz*") on quantum computer.
2. Measure P.
3. Repeat 1 and 2 over and over to estimate $\langle P \rangle$ for each P.
4. Classically combine:

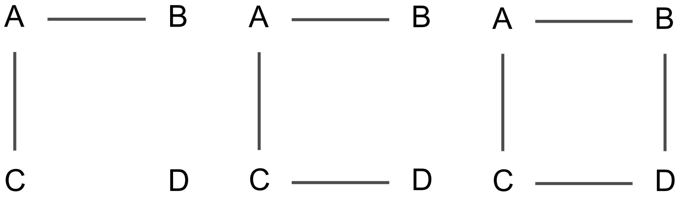
$$\langle H \rangle = \sum_P h_P \langle P \rangle.$$

5. Vary ansatz and return to step 1.

An application of contextuality

Variational quantum eigensolver: characterized by measurements of Pauli operators in Hamiltonian. So...

- Can classify VQE instances by whether or not they exhibit contextuality:
  - ∃ necessary and sufficient condition for contextuality of any set of n-qubit Pauli operators:[3]



Set is contextual iff it contains a subset of four Pauli operators with any of above commutation relations (edge ⇔ commutes).

[3] W. Kirby and P. Love, Phys. Rev. Lett. 123, 200501 (2019).

Extensions and applications

- Complexity of finding ground states of noncontextual Hamiltonians is in NP, rather than QMA.[4].
- ⇒ Screening potential quantum simulation instances.
- Splitting Hamiltonian into noncontextual part to be simulated classically and noncontextual part to be simulated 'quantumly'.[5]
- Whether and to what extent the contextual structure of a molecular Hamiltonian relates to its chemistry (future work for some enterprising person).

[4] W. Kirby and P. Love, Phys. Rev. A 102, 032418 (2020)
[5] W. Kirby, A. Tranter, and P. Love, Quantum 5, 456 (2021).

A limitation of contextuality

Stabilizer subtheory:

- Allowed states: *stabilizer states* = eigenstates of (CCS) Pauli operators.
- Allowed operations: map stabilizer states to stabilizer states.
- Allowed measurements: Pauli measurements.

Gottesman-Knill Theorem[6] (roughly): the stabilizer subtheory can be simulated classically efficiently.

But, stabilizer subtheory contains contextuality: magic square!

$\Rightarrow$ Quantum advantage over classical computation cannot be explained entirely by contextuality.

[6] S. Aaronson and D. Gottesman, Phys. Rev. A 70, 052328 (2004)

Thank you!

Any questions?

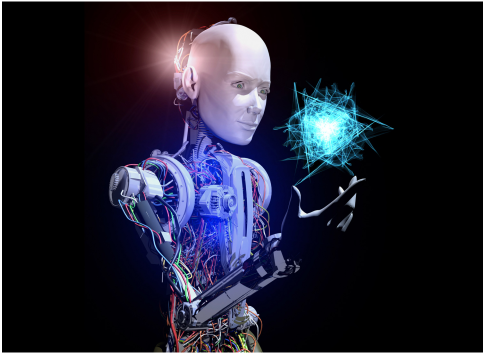


Figure: A quantum computer.

https://networkposting.com/wp-content/uploads/2017/09/quantum-computing-1.jpg